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ON A GENERALIZATION OF TRANSITIVITY FOR DIGRAPHS

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In this paper we investigate the following generalization of transitivity: A digraph D is *(m, n)-transitive* whenever there is a path of length m from x to y there is a subset of $n+1$ vertices of these $m+1$ vertices which contain a path of length n from x to y.

Here we study various properties of (m, n)-transitive digraphs. In particular, *$(m, 1)$-transitive* tournaments are characterized. Their similarities to transitive tournaments are analyzed and discussed.

Various other results pertaining to *$(m, 1)$-transitive* digraphs are given.

Introduction

The study of transitive digraphs and their underlying properties has been for some time a "completed" topic, though new applications and variations occur. It is the purpose of this paper to discuss a generalization of transitivity for digraphs. For a digraph D, we denote by $V(D)$ and $A(D)$ the vertex and arc set respectively. For convenience V and A will be used when no confusion results. Let $U \subseteq V(D)$, then the subdigraph $\langle U\rangle$ of D *induced* by U is the subdigraph of D with $V(\langle U\rangle) = U$ and $A(\langle U\rangle)$ consisting of all the arcs of D joining vertices of U. A *path* of D is formed by a sequence $x_0, x_1, x_2, \ldots, x_m$ of vertices, all distinct, such that for $i = 1, 2, \ldots, m$, $x_{i-1}x_i \in A(D)$. The *length* of a path is the number of arcs it contains. We will refer to such a path as an x_0–x_m path of length m or alternatively a path of length m from x_0 to x_m. A *cycle* is an x_0–x_m path together with arc x_mx_0. A subdigraph D' of D is *strongly connected* if for each pair x, y of distinct vertices in D', D' contains an x–y path and a y–x path. A *strong component* in D is a maximal strongly connected subdigraph. Finally, a *tournament* is a digraph in which, for each pair x, y of distinct vertices, exactly one of the arcs xy and yx is in $A(D)$.

We now introduce a generalization of transitivity presented by Harary through McMorris. A digraph D is *(m, n)-transitive* if whenever there is an x_0–x_m path of

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length m there is a subset of $n+1$ vertices (of the path), including x_0 and x_m, which induces a digraph containing an x_0–x_m path of length n. Note that the usual transitivity is (2, 1)-transitivity in this notation.

In this paper we consider this generalization and various implications. Our main result characterizes $(m, 1)$-transitive tournaments. We go on to discuss in detail (3, 1)-transitive tournaments, how many there are as well as their structure displayed. We then discuss relations between such transitive digraphs. For example we show that a tournament is (3, 1)-transitive if and only if it is (3, 2)-transitive but that this is not the case for $(m, 1)$ transitivity with $m \geqslant 4$. Finally, we discuss various properties of (m, n)-transitive digraphs and conclude with a number of problems and directions of pursuit.

On $(m, 1)$-transitive tournaments

It is well known that (2, 1)-transitive (or just transitive) asymmetric digraphs are acyclic. If one considers this as not containing a cycle of length 3 or more then in this context $(m, 1)$-transitivity is a direct generalization of transitivity, at least for tournaments. We exhibit this in the following theorem.

Theorem 1. *A tournament T is $(m, 1)$-transitive if and only if it contains no cycles of length $m+1$ or more.*

Proof. Clearly, if T contains no cycles of length $m+1$, then T must be $(m, 1)$-transitive, since every $x-y$ path of length m implies $yx \notin A$, so that $xy \in A$.

Now suppose T is $(m, 1)$-transitive and does contain a cycle of length greater than m. Clearly, T must contain a cycle of length greater than $m+1$.

First, suppose there are no cycles of length $m+2$, i.e., the shortest cycle greater than $m+1$ is of length $k \geq m+3$. Denote the vertices $x_0x_1x_2, \ldots, x_{m+2}, x_{m+3}, \ldots, x_k$. It follows then that $x_{i+2}x_i$ and $x_{i+3}x_i \in A$ for $i = 0, 1, \ldots, k$, where the addition of subscripts is taken modulo $(k+1)$.

Consider the following x_m–x_0 path of length m:

$$x_mx_{m-2}x_{m-1}x_{m-3}x_{m-5}x_{m-4}x_{m-6}x_{m-8} \ldots x_3x_1x_2x_0, \quad \text{if } m \equiv 0 \pmod 3,$$

$$x_mx_{m+2}x_{m-2}x_{m-1}x_{m-4}x_{m-6}x_{m-5}x_{m-7} \ldots x_3x_1x_2x_0, \quad \text{if } m \equiv 1 \pmod 3,$$

$$x_mx_{m+1}x_{m-2}x_{m-4}x_{m-3}x_{m-5}x_{m-7}x_{m-6} \ldots x_3x_1x_2x_0, \quad \text{if } m \equiv 2 \pmod 3.$$

Since T is $(m, 1)$-transitive, this implies $x_mx_0 \in A$, but this is a contradiction since $x_0x_1 \ldots x_m$ is a path of length m from x_0 to x_m, implying $x_0x_m \in A$.

Hence we may assume there is a cycle of length $m+2$, say $x_0x_1 \ldots x_{m+1}x_0$. It follows that $x_{i+2}x_i \in A$ for $i = 0, 1, \ldots, m+1$ where addition of subscripts is taken modulo $(m+2)$. If for some i, both x_ix_{i+3} and $x_{i+1}x_{i+4} \in A$, then

$$x_0x_1 \ldots x_ix_{i+3}x_{i+1}x_{i+4} \ldots x_{m+1}$$

would be a path of length m from x_0 to x_{m+1} which would contradict $x_{m+1}x_0 \in A$ since T is $(m, 1)$-transitive. Thus, no such i exists. As above if $m \equiv 0$ or $2 \pmod 3$ a contradiction is quickly established by considering the given paths of length m. In the case $m \equiv 1 \pmod 3$, since there does not exist i with both x_ix_{i+3} and $x_{i+1}x_{i+4} \in A$, it must be the case that there exists an i with $x_{i+1}x_{i-2}$ and $x_{i+3}x_i$ both elements of A. By relabeling $i = m - 2$ and the $(m-2)$-cycle appropriately, the $x_m - x_0$ path of length m given in the $m \equiv 1 \pmod 3$ case above again yields a contradiction. With all cases exhausted, the proof is complete. □

This theorem gives a complete characterization of $(m, 1)$-transitive tournaments. For asymmetric $(m, 1)$-transitive digraphs in general, restrictions on cycle length are much more subtle. While it is well known that $(2, 1)$-transitivity implies no cycle of length three or more, and we prove a similar result when $m = 3$, no such restriction exists if $m \geq 4$. Examples are given in Fig. 1 and the construction following it.

The structure and existence of long cycles in $(m, 1)$-transitive digraphs has been considered in [2]. It is easy to see that the following digraph is $(m, 1)$-transitive and contains a cycle of length $2m - 2$ for $m \geq 4$. The vertices are $x_0, x_1, x_2, \ldots, x_{2m-3}$ and the edges are x_ix_{i+1} and x_ix_{i+m} for $i = 0, 1, \ldots, 2m-3$ (subscript addition taken modulo $(2m-2)$). Fig. 1 shows the case $m = 4$.

Proposition 2. *If D is a $(3, 1)$-transitive asymmetric digraph, then D contains no cycles of length 4 or more.*

Proof. Suppose the result is false and let C be a shortest cycle of length 4 or more. Clearly, since D is $(3, 1)$-transitive it can contain no cycles of length 4 and if C was a cycle of length 6 or more a shorter such cycle would result by the presence of arc x_0x_3. Hence, we may assume C is a cycle of length 5. Label the vertices of C: x_0, x_1, x_2, x_3 and x_4. Since D is $(3, 1)$-transitive, x_0x_3 and $x_4x_2 \in A$. By considering the paths $x_2x_3x_4x_0$ and $x_0x_3x_4x_2$ a contradiction results. Thus D can contain no cycles of length 4 or more. □

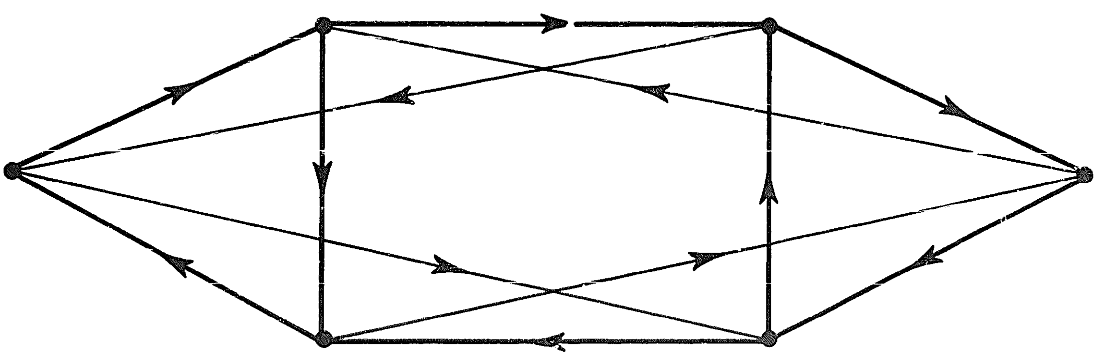

Fig. 1. A (4, 1)-transitive digraph with a cycle of length 6.

Next we use Theorem 1 to count the number of $(m, 1)$-transitive tournaments.

Let $S_1, S_2, \ldots, S_n$ be the strong components of D. The condensation D^* of D is the digraph with vertex set $\{S_1, S_2, \ldots, S_n\}$ and S_iS_j is an arc of D^* if and only if $i \neq j$ and for some vertex $u_i \in S_i$ and $u_j \in S_j$, $u_iu_j \in A(D)$. It is well known that in the case of a tournament T, T^* is transitive. Also, for a tournament T, any strong component is Hamiltonian, i.e., there is a directed cycle containing each vertex of the component. Using these facts, we can more accurately describe the structure of $(m, 1)$-transitive tournaments.

For convenience, we will call the order of D^* the height of D and denote it by $h(D)$. The concept of height is exhibited in Fig. 2.

By the previous observations and by Theorem 1, if T is $(m, 1)$-transitive, then S_i is a Hamiltonian tournament of order at most m. Furthermore, to construct an $(m, 1)$-transitive tournament of height h, we can select any h strong (i.e., Hamiltonian) tournaments and give the structure of T^* as shown and the resulting tournament is $(m, 1)$-transitive.

Moon [4] determined the number of strong tournaments. If we let $t(m)$ be the number of strong tournaments of order m or less than the following result holds.

Theorem 3. *There are $(t(m))^h$ $(m, 1)$-transitive tournaments of height h.*

To count the number of $(m, 1)$-transitive tournaments of a particular order is a bit more difficult since it would include the number of partitions of n such that no part is 2. For the case of $(3, 1)$-transitive tournaments the problem is somewhat less difficult since there are only 2 possible strong components for T^*; a single vertex or a 3-cycle. In this case we get the following result.

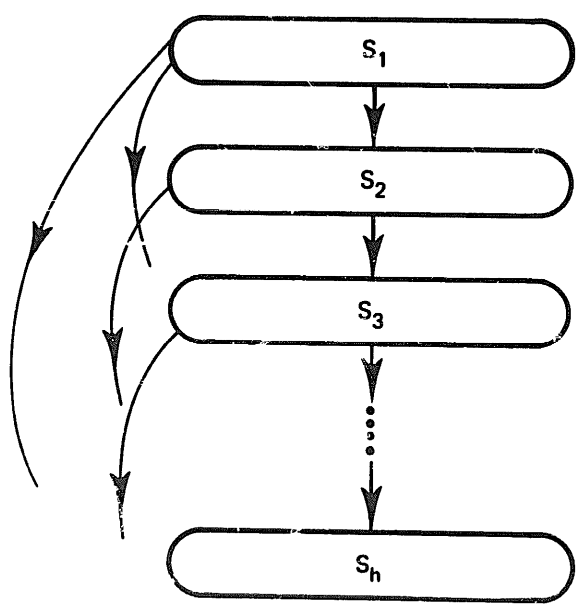

Fig. 2. The transitive tournament T^*, the condensation of a tournament T.

Theorem 4. *Let n be a positive integer and r be the largest integer $\leq \frac{1}{3}n$. The number of* (3, 1)*-transitive tournaments of order n is*

$$\sum_{i=0}^{r} \binom{n-2i}{i}.$$

Proof. At most $\frac{1}{3}n$ of the $h(T)$ strong components are triangles. If i triangles occur, T has height $n-2i$. Thus, there are $\binom{n-2i}{i}$ (3, 1)-transitive tournaments of order n and height $n-2i$. □

We note that the degree sets and sequences for (3, 1)-transitive tournaments are studied in [2].

On relations between transitivities

In this section we consider the connection between tournaments that are $(m, 1)$-transitive and (m, i)-transitive. There are some obvious relations between transitivities of digraphs. For example, $(m, 1)$-transitivity implies $(t(m-1)+1, 1)$-transitivity, $(m+t, 1+t)$-transitivity and (tm, t)-transitivity for all integers $t \geq 1$. In tournaments, $(m, 1)$-transitivity implies $(n, 1)$-transitivity for $n \geq m$ according to Theorem 1, and there are connections between $(m, 1)$-transitivity and (m, i)-transitivity as the next theorem shows:

Theorem 5. *If a tournament T is $(m, 1)$-transitive, then T is (m, k)-transitive for $k = 1, 2, \ldots, m$.*

Proof. Let $x_0, x_1, \ldots, x_m$ be an m-path in a $(m, 1)$-transitive tournament T and let k be an integer, $1 \leq k \leq m$. The tournament T' induced by $\{x_0, x_1, \ldots, x_m\}$ in T is clearly not Hamiltonian; it has a directed cut, i.e., $V(T') = V(T_1) \cup V(T_2)$, where $V(T_1) \cap V(T_2) = \emptyset$ and y_1y_2 is an arc of T' for all $y_1 \in V(T_1)$, $y_2 \in V(T_2)$. Let $p = |V(T_1)|$. Since $x_0x_1 \ldots x_m$ is a (Hamiltonian) path of T', $\{x_0, x_1, \ldots, x_{p-1}\} = V(T_1)$ and $\{x_p, \ldots, x_m\} = V(T_2)$ follow.

To see that T is (m, k)-transitive, we exhibit a suitable k-path P_k from x_0 to x_m:

$$P_k = \begin{cases} x_0x_1 \ldots x_kx_m, & \text{if } k \leq p-1, \\ x_0x_1 \ldots x_{p-1}x_{q-1}x_{q+1} \ldots x_m, & \text{if } k < p-1, \end{cases}$$

where $q = m - k + p - 1$. □

In case $m = 3$, a stronger result holds.

Theorem 6. *A tournament T is* (3, 1)*-transitive if and only if T is* (3, 2)*-transitive.*

Proof. Suppose T is (3, 2)-transitive but not (3, 1)-transitive. Then it must be the

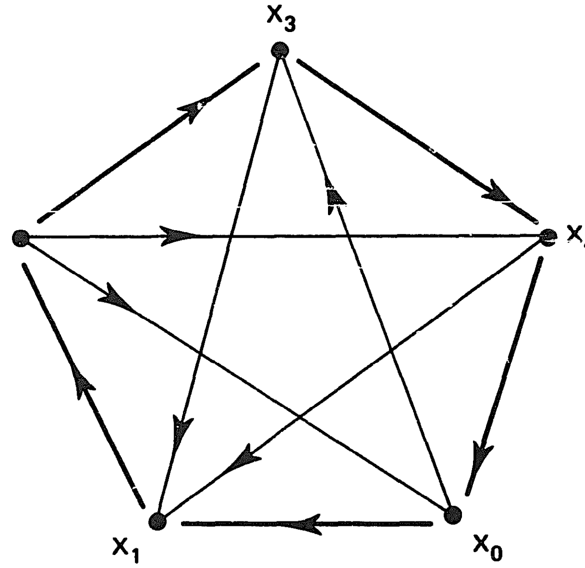

Fig. 3. (4, 2)-transitive but not (4, 1)-transitive tournament.

case that T contains a 4-cycle, say $x_0x_1x_2x_3$. Since T is (3, 2)-transitive either x_1x_3 or $x_0x_2 \in A(T)$. Without loss of generality, suppose $x_0x_2 \in A(T)$. By considering the path $x_2x_3x_0x_1$ it must be the case that $x_3x_1 \in A(T)$. Now consider the path $x_1x_2x_3x_0$; a contradiction results since neither x_2x_0 nor x_1x_3 can be an arc of T. Thus T can contain no cycles of length 4 and therefore T is (3, 1)-transitive. □

Generally (m, k)-transitivity does not imply $(m, 1)$-transitivity for tournaments. In fact, the only case when it does appears in Theorem 6. For $k = 2$ and $m = 4$ the following tournament demonstrates our assertion. The tournament of Fig. 3 is not (4, 1)-transitive since it is Hamiltonian. There is a 2-path from x_i to x_j if $i \neq j$, $0 \leq i \leq 4$, $0 \leq j \leq 4$, except for the following pairs: $i = 0$, $j = 3$; $i = 1$, $j = 2$; $i = 3$, $j = 4$; $i = 4$, $j = 0$. However, for these exceptional pairs (i, j), there are no 4-paths from x_i to x_j. Therefore the tournament is (4, 2)-transitive.

If $k = 2$ and $m \geq 4$, the construction of $(m, 2)$-transitive tournaments that are not $(m, 1)$-transitive is very complex and will appear in [2].

Conclusion

This generalization of transitivity seems like a fruitful area of research. There seems to be a number of directions to be pursued.

(1) Completely characterize the cycle structure of $(m, 1)$-transitive digraphs.

(2) Study the cycle structure, or maximum cycle length of (m, n)-transitive digraphs, and/or tournaments.

(3) Consider the problem of what graphs can be given (m, n)-transitive orientations?

(4) Is there a reasonable way to define and study the (m, n)-transitive closure of a digraph?

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